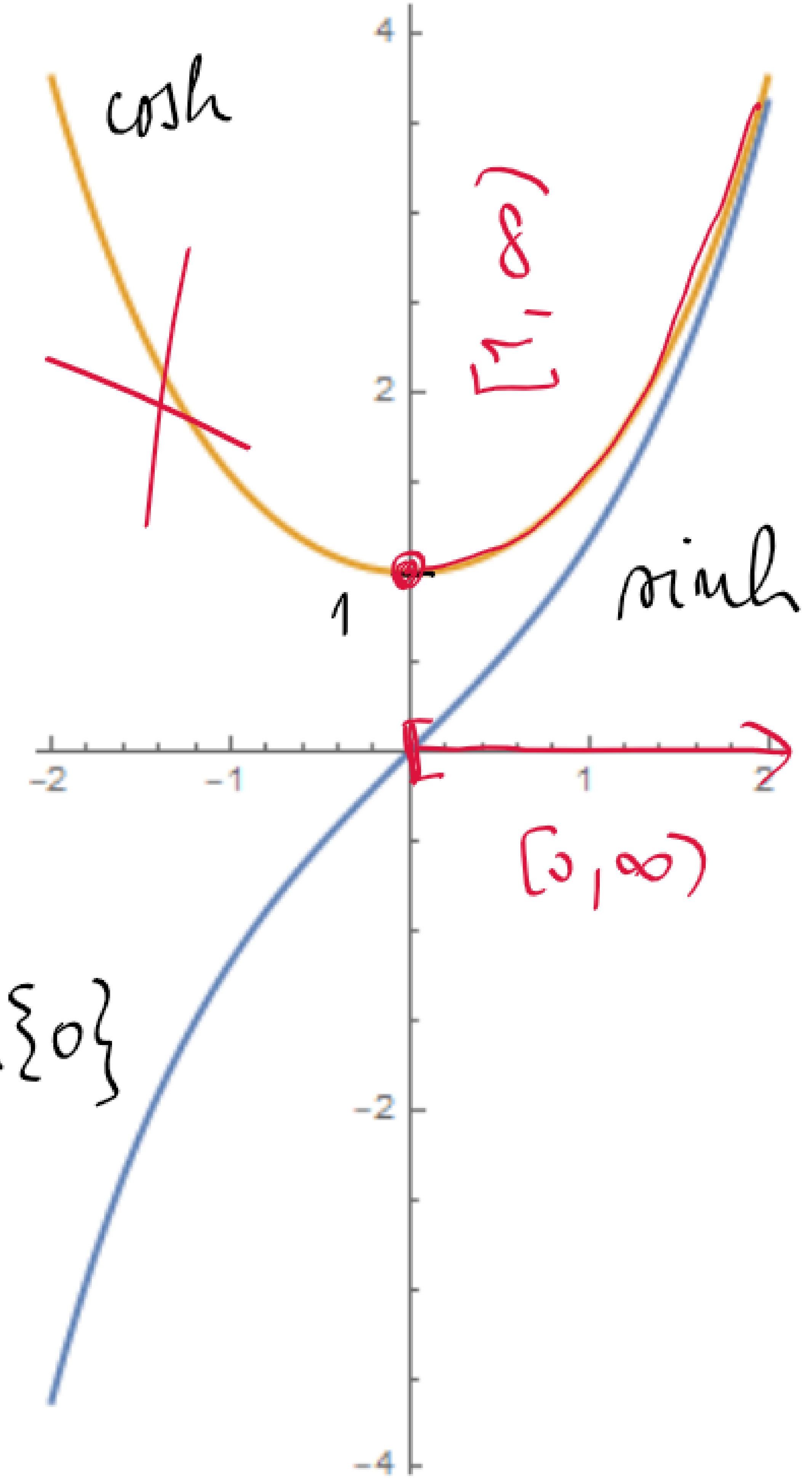
Připomenutí: $\operatorname{sh} x = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\operatorname{ch} x = \cosh x = \frac{e^x + e^{-x}}{2}$$

- $\operatorname{sh}' = \operatorname{ch}$, $\operatorname{ch}' = \operatorname{sh}$
- $\operatorname{ch}^2 - \operatorname{sh}^2 = 1$



Dále definujeme

$\operatorname{tgh} x = \dfrac{\operatorname{sh} x}{\operatorname{ch} x}$ $\quad D_{\operatorname{tgh}} = \mathbb{R}$

$\operatorname{cotgh} x = \dfrac{\operatorname{ch} x}{\operatorname{sh} x}$ $\quad D_{\operatorname{cotgh}} = \mathbb{R} \setminus \{0\}$

- $\operatorname{sh} : \mathbb{R} \xrightarrow{\text{na}} \mathbb{R}$ bijekce

  $\operatorname{ch} : \mathbb{R} \xrightarrow{\text{na}} [1,\infty)$

Derivace tgh, cotgh:

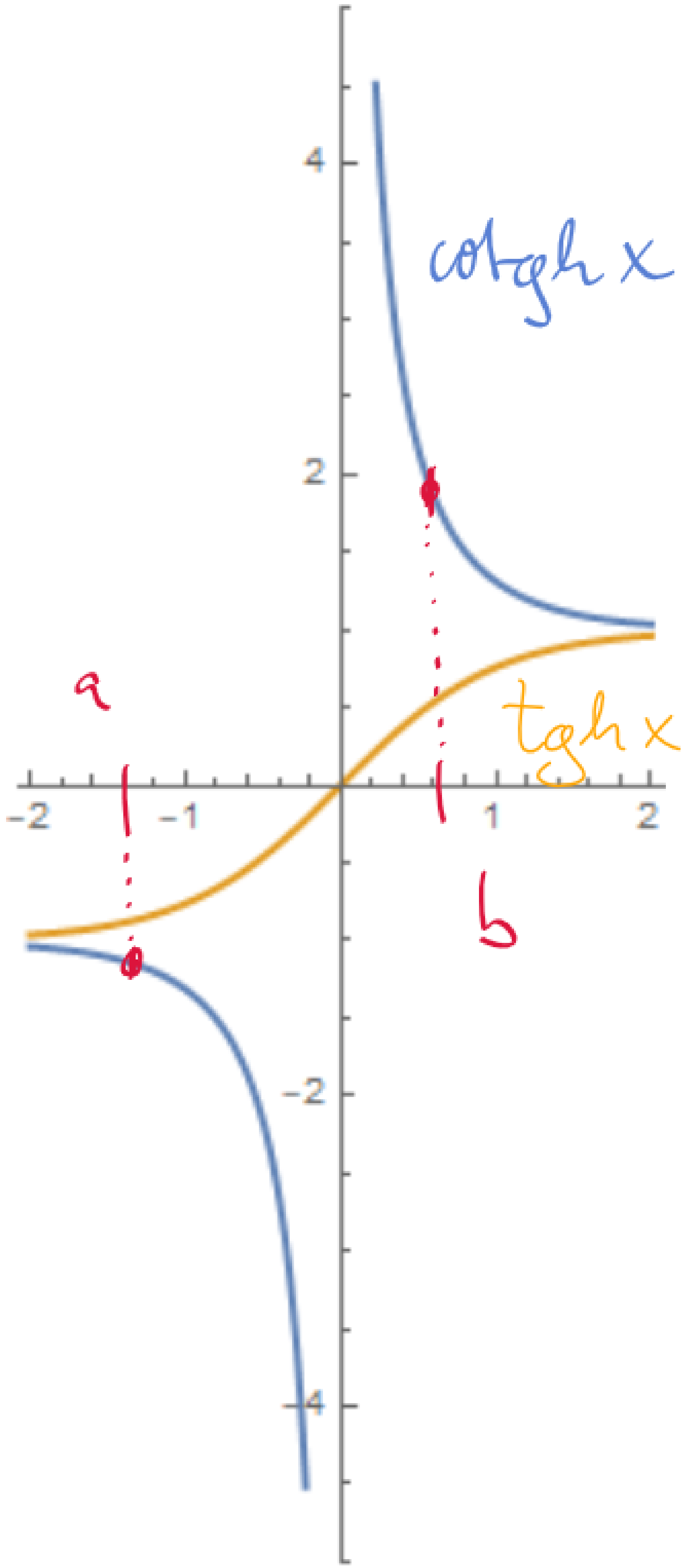
- $(\operatorname{tgh} x)' = \left(\dfrac{\operatorname{sh} x}{\operatorname{ch} x}\right)' =$

$$= \frac{\operatorname{ch} x \cdot \operatorname{ch} x - \operatorname{sh} x \cdot \operatorname{sh} x}{\operatorname{ch}^2 x} =$$

$$= \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x}$$

$$\left[(\operatorname{tg} x)' = \frac{1}{\cos^2 x} \right]$$

- $(\operatorname{cotgh} x)' = \dfrac{-1}{\operatorname{sh}^2 x}$



Pozn.: cotgh není klesající na $D_{\operatorname{cotgh}} = (-\infty,0) \cup (0,\infty)$, ačkoliv je klesající na $(-\infty,0)$ i na $(0,\infty)$.

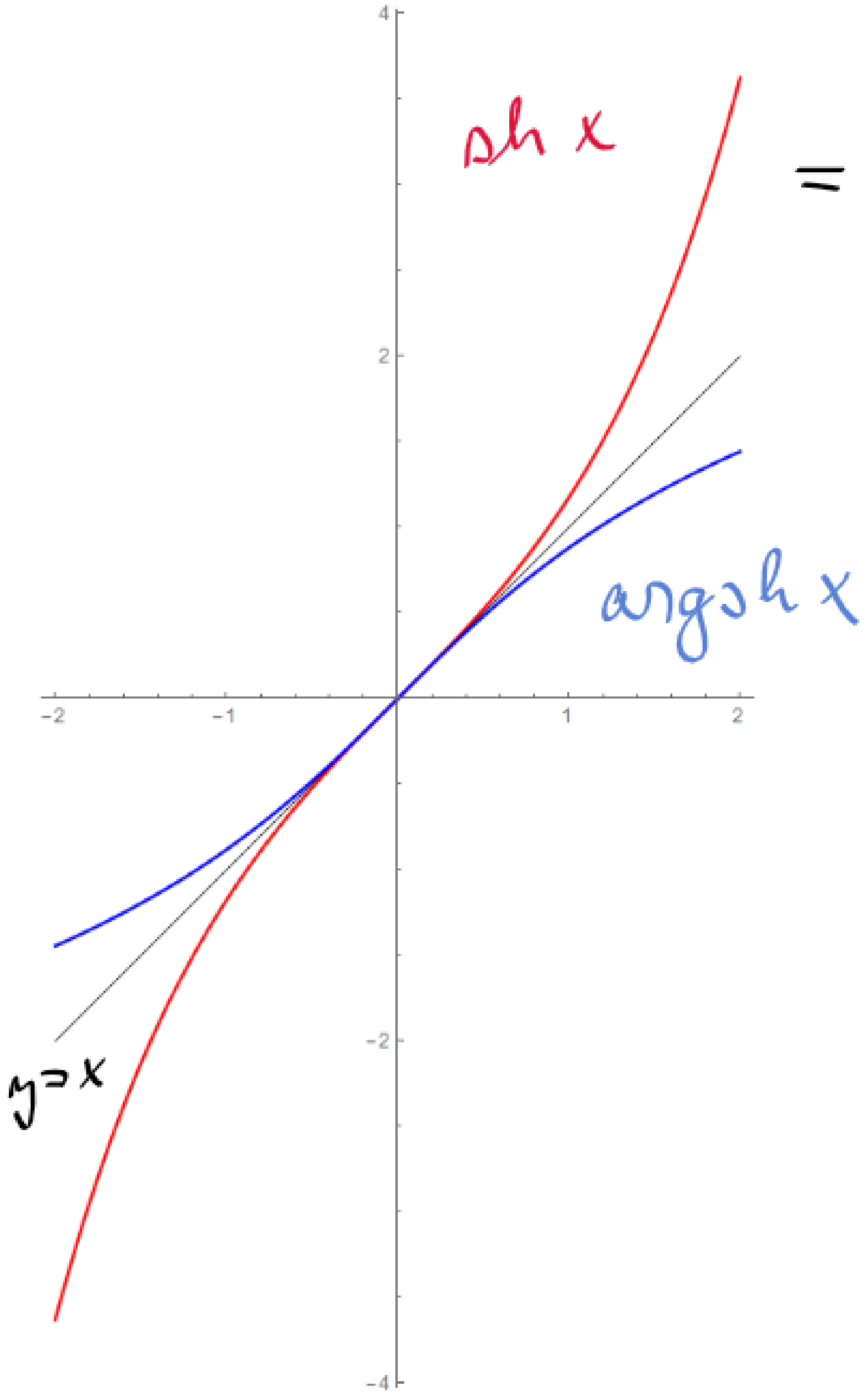
Hyperbolometrické funkce

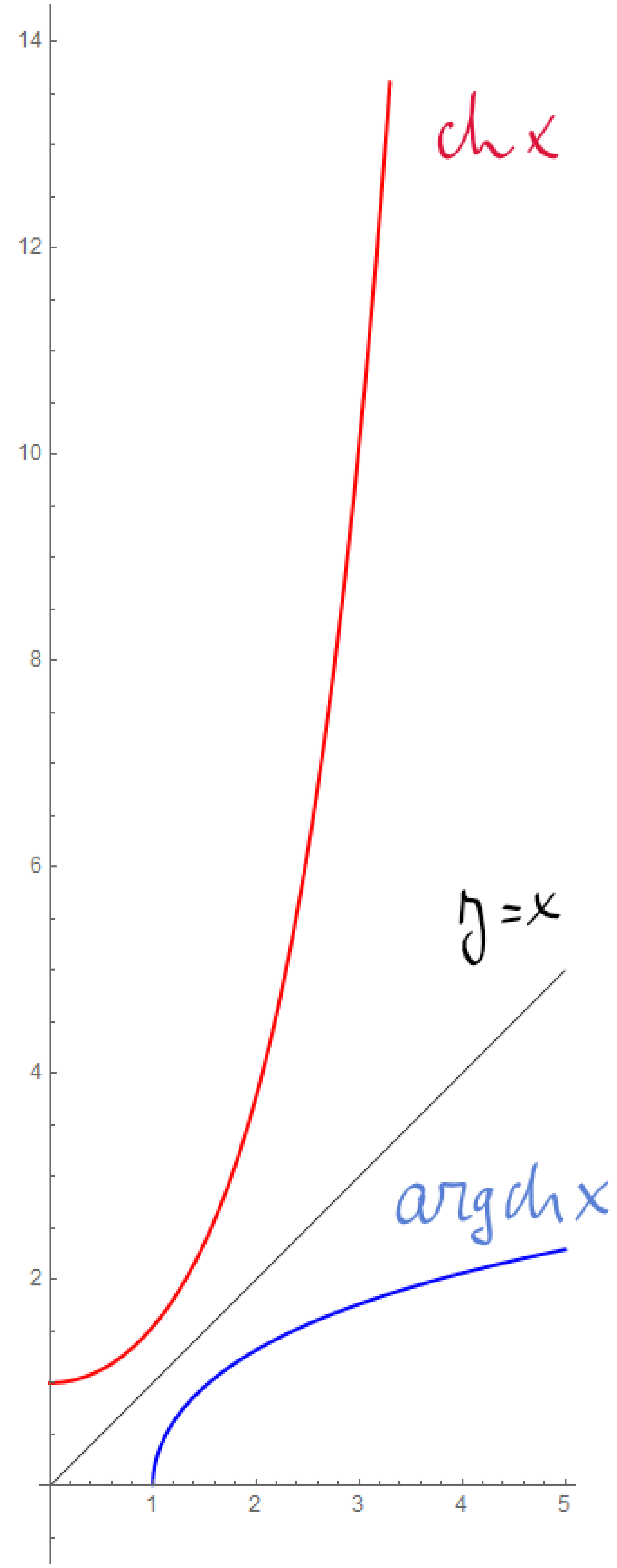
Víme:

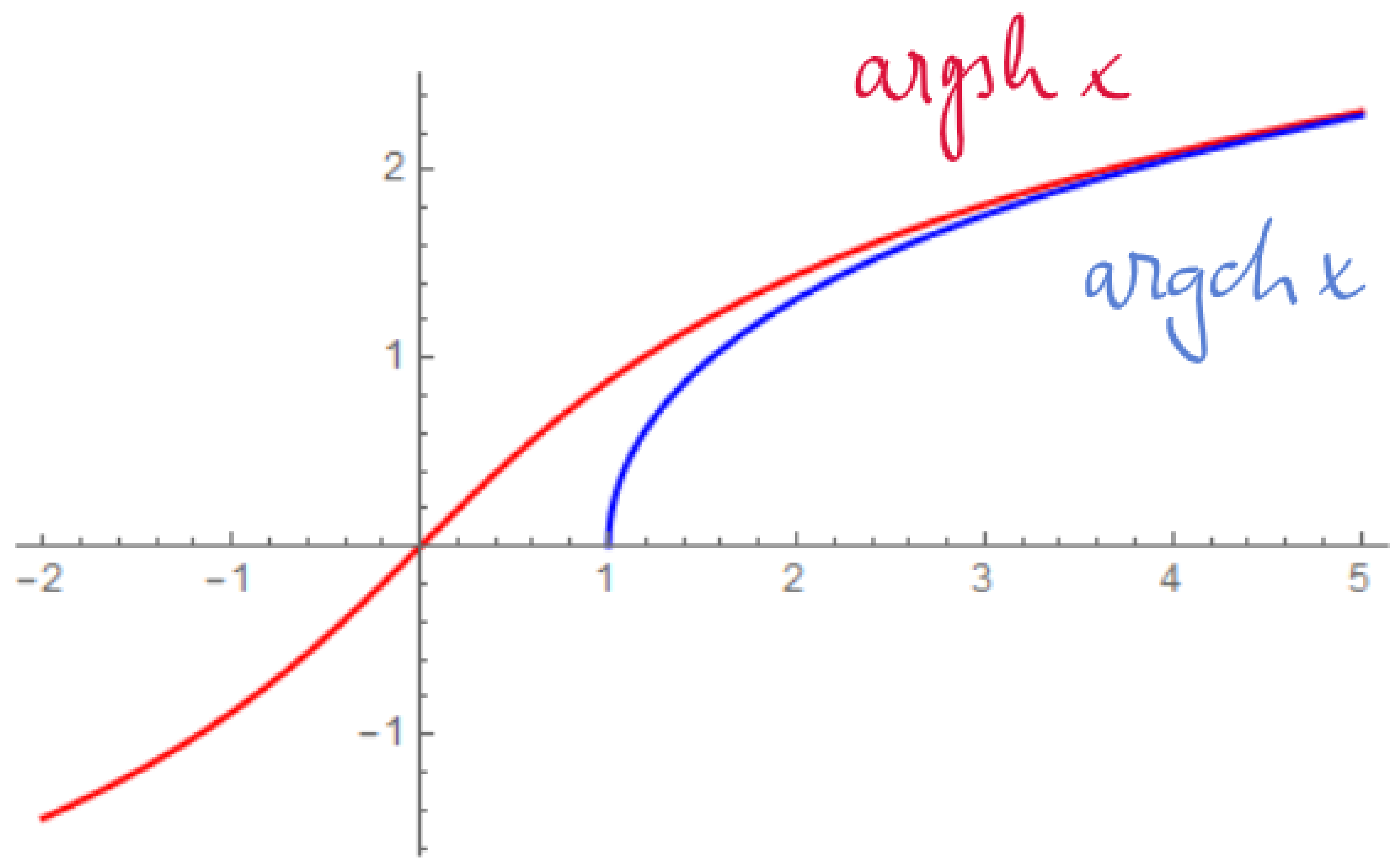
$$\sinh^{-1} x = \operatorname{argsh} x = \ln\left(x + \sqrt{1+x^2}\right), \quad x \in \mathbb{R}$$

Dále platí:

$$\cosh^{-1} x = \operatorname{argch} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1, \infty)$$







Připomenutí:

- $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$
- $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$
- $(\operatorname{arccotg} x)' = \dfrac{-1}{1+x^2}$

• $(\operatorname{argsh} x)' = ?$

$$L = \left(\ln\left(x+\sqrt{1+x^2}\right)\right)' = \frac{1}{x+\sqrt{1+x^2}} \cdot$$

$$\cdot \left(x+\sqrt{1+x^2}\right)' = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right) =$$

$$= \frac{\cancel{\sqrt{1+x^2}+x}}{\cancel{\left(x+\sqrt{1+x^2}\right)}\cdot\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{x^2+1}}$$

• $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$ • $(\operatorname{argsh} x)' = \dfrac{1}{\sqrt{1+x^2}}$

• $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \quad x \in (-1,1)$

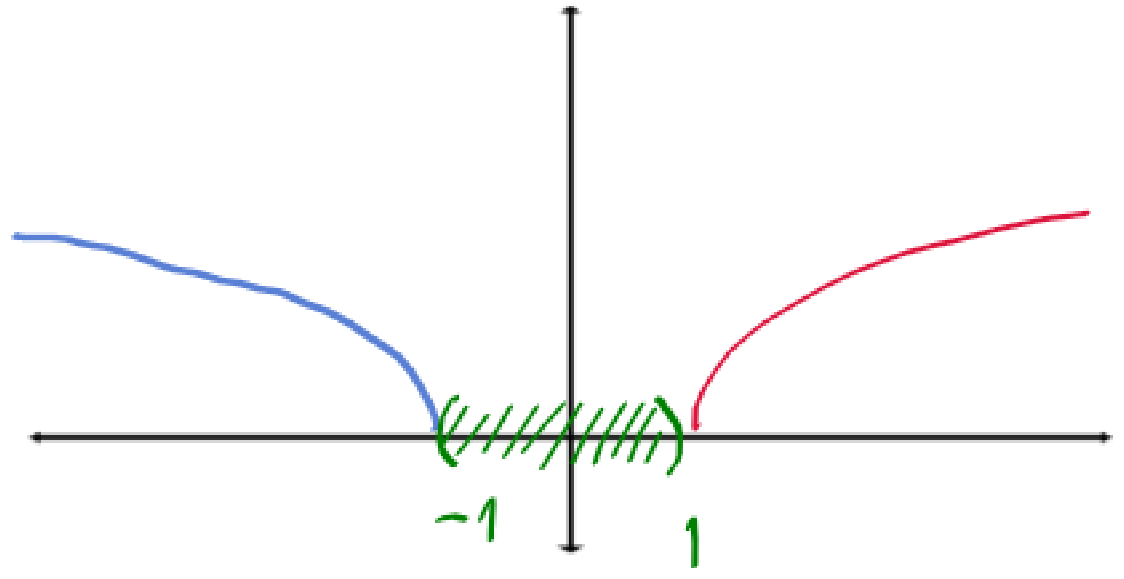
• $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{argsh} x + C = \ln\left(x+\sqrt{1+x^2}\right),$

$x \in \mathbb{R}$

• $(\operatorname{argch} x)' = \left(\ln\left(x+\sqrt{x^2-1}\right)\right)' =$

$$= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1+\frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{\sqrt{x^2-1}},$$

$x \in (1,\infty)$



• Analog $(\operatorname{argch}(-x))'$

$$= \frac{1}{\sqrt{(-x)^2-1}} \cdot (-1) = \frac{-1}{\sqrt{x^2-1}}$$

• Celkem: $\boxed{(\operatorname{argch}|x|)' = \dfrac{\operatorname{sgn} x}{\sqrt{x^2-1}}}$,

$x \in (-\infty,-1) \cup (1,\infty)$.

$$\int \frac{\operatorname{sgn} x}{\sqrt{x^2-1}}\,dx = \operatorname{argch}|x| + C, \quad x \in (-\infty,-1),\ (1,\infty)$$

Tj. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{sgn} x \cdot \operatorname{argch}|x| + C, \quad x \in$ –||–

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

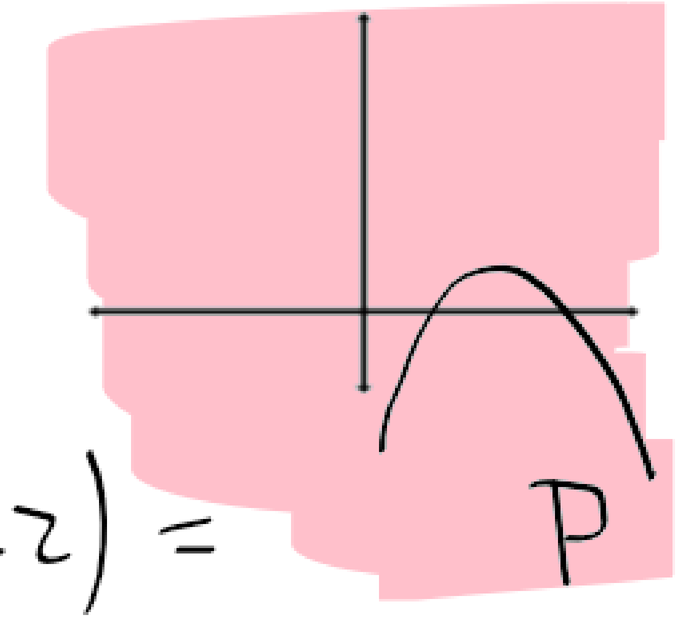
$$= \frac{4}{3}\int \frac{dx}{\left(\sqrt{\frac{4}{3}}\,x + \sqrt{\frac{4}{3}}\cdot\frac{1}{2}\right)^2+1}$$

... substituce ... $C\int \frac{dy}{1+y^2}$ ...

Integrály tvaru $\displaystyle\int \frac{1}{\sqrt{P(x)}}\,dx = \int \frac{dx}{\sqrt{ax^2+bx+c}}$

kde P je polynom 2. stupně

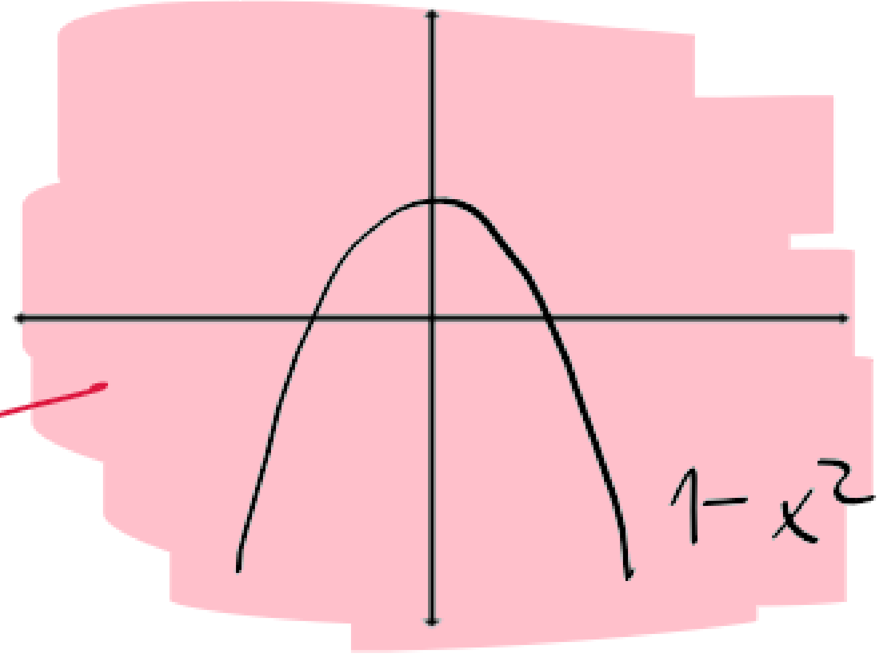
nyní umíme všechny.



- 1. PŘÍPAD: $P(x) = -(x-1)(x-2) =$
  $= -(x^2-3x+2)$

Počítejme $\displaystyle\int \frac{1}{\sqrt{3x-2-x^2}}\,dx$ ... na arcsin

VÍME: $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$



$$3x-2-x^2 = -(x^2-3x+2) =$$
$$= -\left(\left(x-\frac{3}{2}\right)^2+2-\frac{9}{4}\right) =$$
$$= \frac{1}{4} - \left(x-\frac{3}{2}\right)^2 = \frac{1}{4}\left(1-(2x-3)^2\right)$$

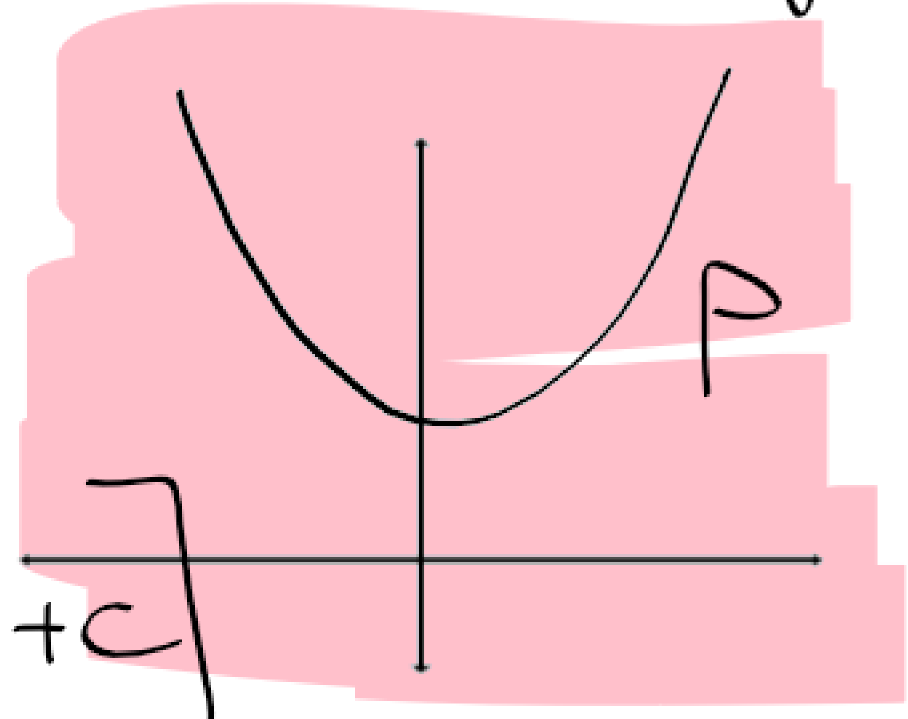
$$\int \frac{1}{\sqrt{3x-2-x^2}}\,dx = 2\int \frac{dx}{\sqrt{1-(2x-3)^2}} = \left| \begin{array}{l} y = 2x-3 \\ dy = 2dx \end{array} \right| =$$

$$= \int \frac{dy}{\sqrt{1-y^2}} = \arcsin y + C = \arcsin(2x-3) + C$$

2. PŘÍPAD: P nemá kořeny, $P > 0$.

$P(x) = x^2 + 2x + 2$ ... nemá kořeny

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \Bigg|$$ převedeme na argsh: $\left[\int \frac{1}{\sqrt{1+x^2}} = \operatorname{argsh} x + C\right]$



$$x^2+2x+2 = (x+1)^2 + 1$$

$$= \int \frac{dx}{\sqrt{(x+1)^2+1}} = \left| \begin{matrix} y = x+1 \\ dy = dx \end{matrix} \right| = \int \frac{dy}{\sqrt{y^2+1}} =$$

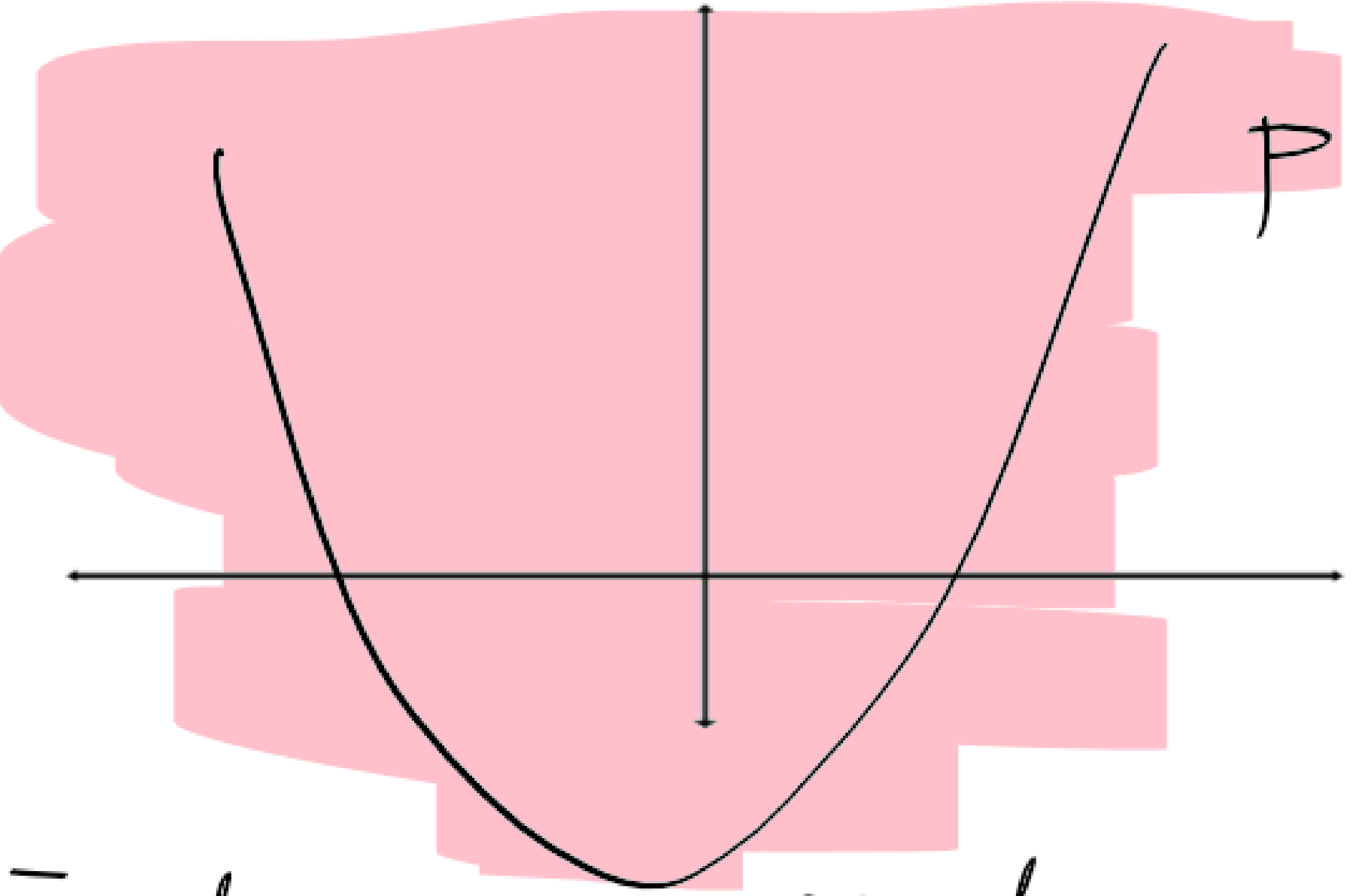
$$= \operatorname{argsh} y + C = \operatorname{argsh}(x+1) + C =$$

$$= \ln\left((x+1) + \sqrt{(x+1)^2+1}\right), \quad x \in \mathbb{R}.$$

3. PŘÍPAD: P má 2 různé kořeny

$$P(x) = x^2 - 2x - 3$$

$$= (x+1)(x-3).$$



$\int \frac{1}{\sqrt{x^2-2x-3}}$ ... převede se na argch x.

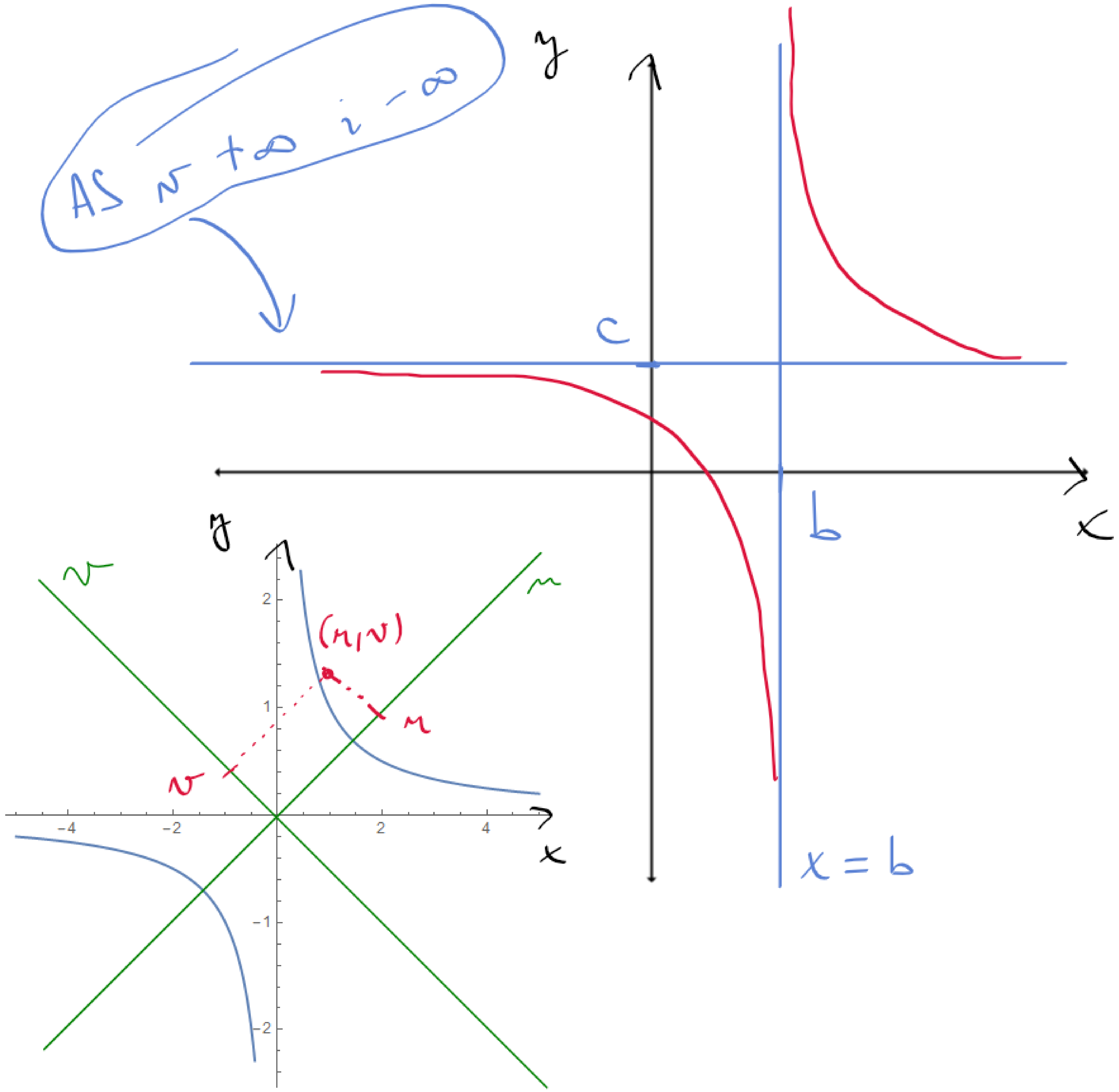
4. PŘÍPAD: $P < 0$ $(x \in \mathbb{R})$

$\frac{1}{\sqrt{P(x)}}$ není def. pro žádné $x \in \mathbb{R}$.

Pozn.: Dotek s osou x, např. $P = (x-1)^2$.
Pak 1 je jediný (2-nás) kořen. | $\sqrt{P(x)} = |x-1|$

Hyperboly: • Grafy funkcí:

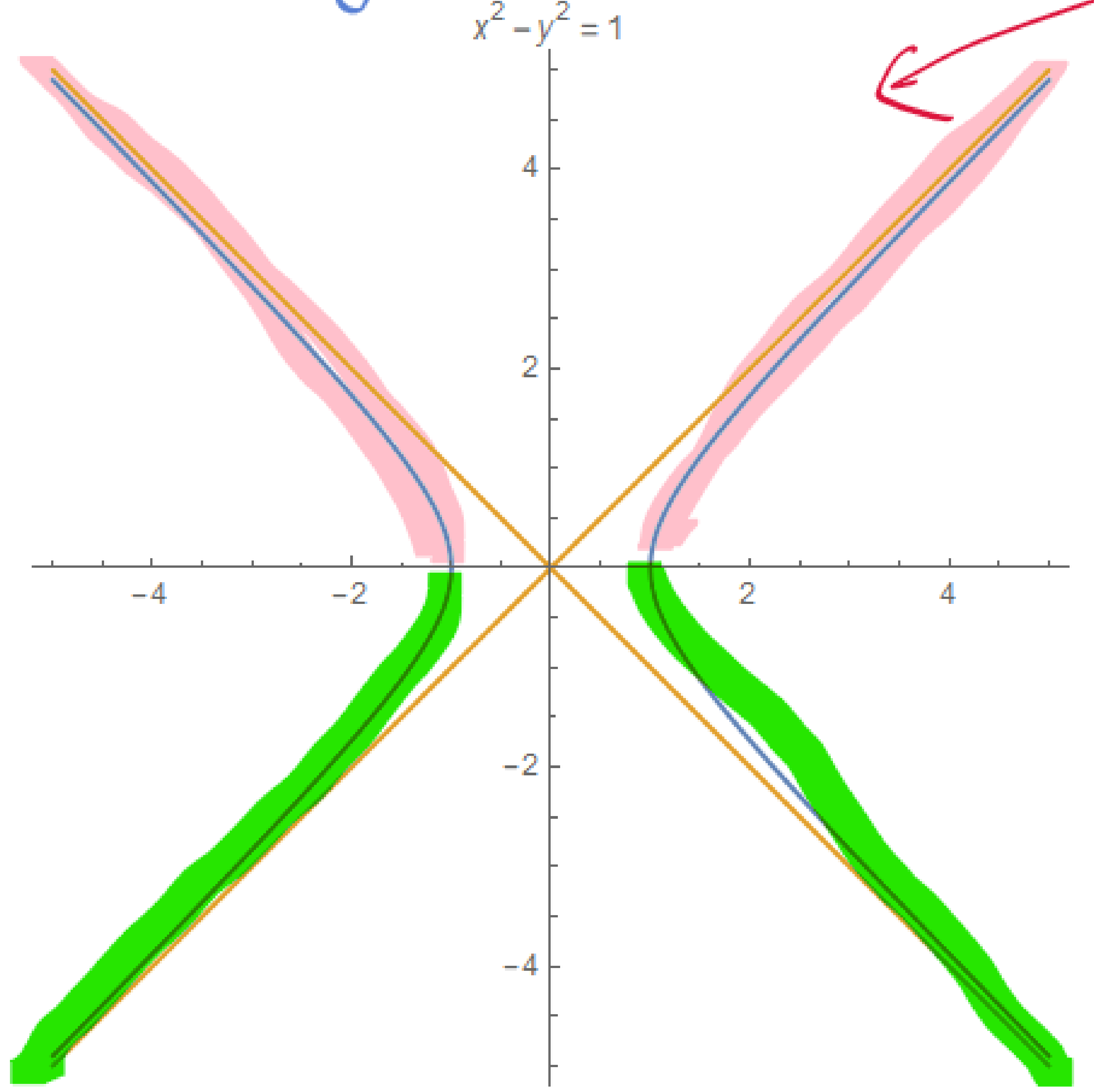
$y = \frac{a}{x}$, resp: $y = \frac{a}{x-b} + c$

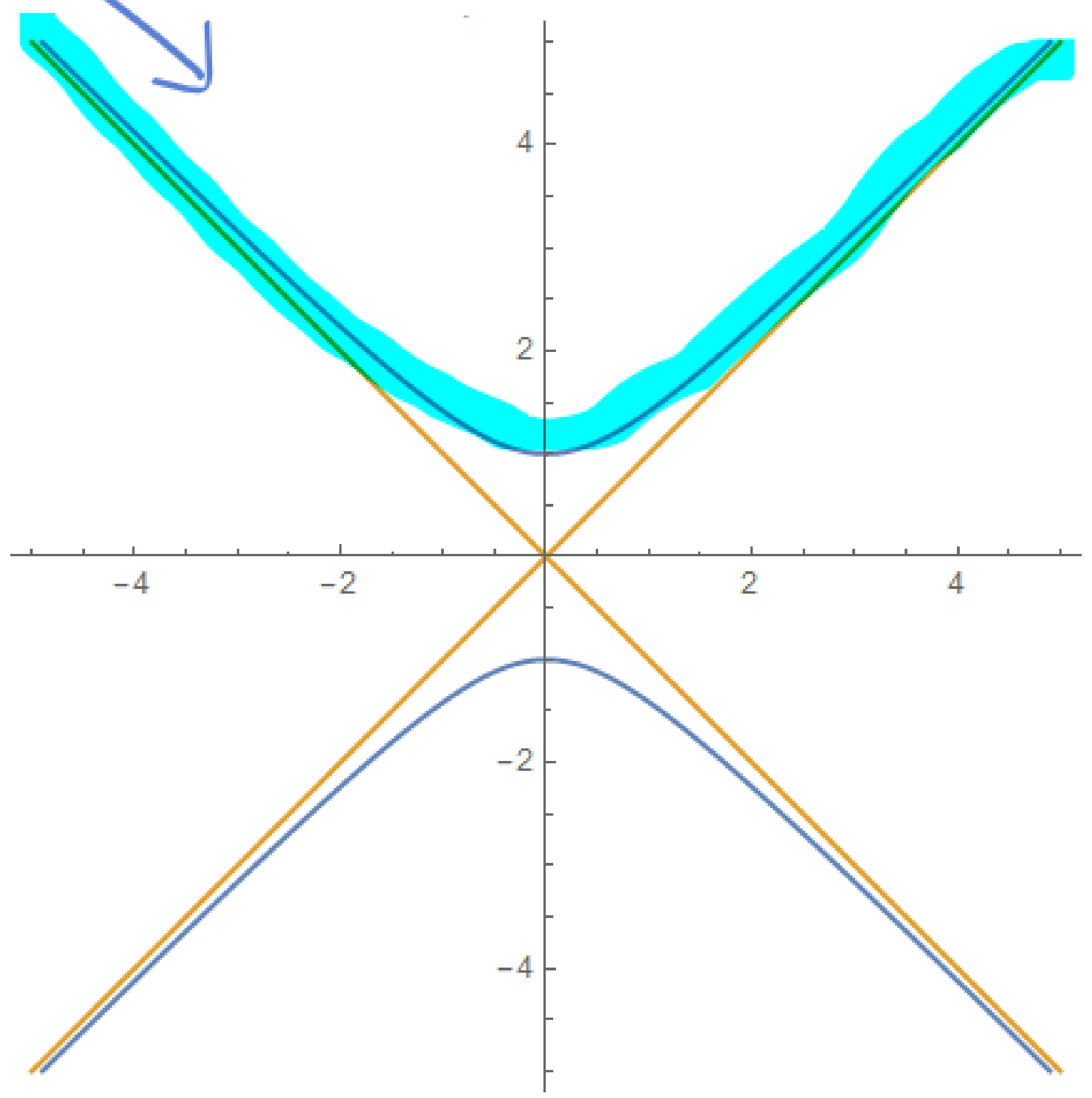


• Implicitní zadání, tj. jako množina bodů splňujících rovnici

$y^2 - x^2 = 1$ resp. $x^2 - y^2 = 1$

$y = \pm\sqrt{1+x^2}$ $\qquad$ $y = \pm\sqrt{x^2-1}$





$f(x) = y = \sqrt{1+x^2}$ ... ASYMPTOTA $y = ax + b$

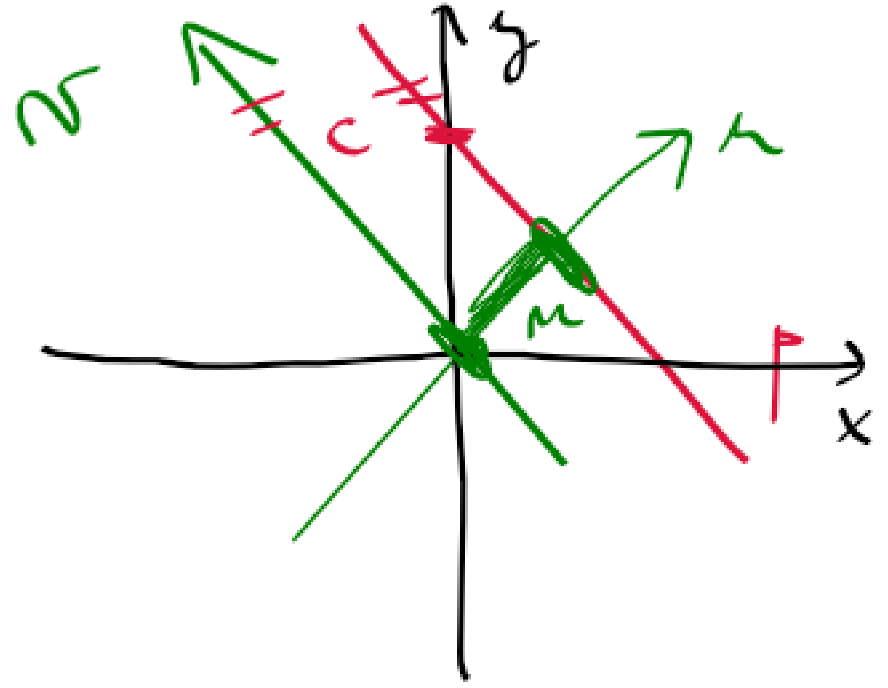
$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = 1$

$b = \lim_{x\to\infty} (f(x) - ax) = \lim_{x\to\infty} \left(\sqrt{1+x^2} - x\right) = 0$

AS. v ∞
$y = x$

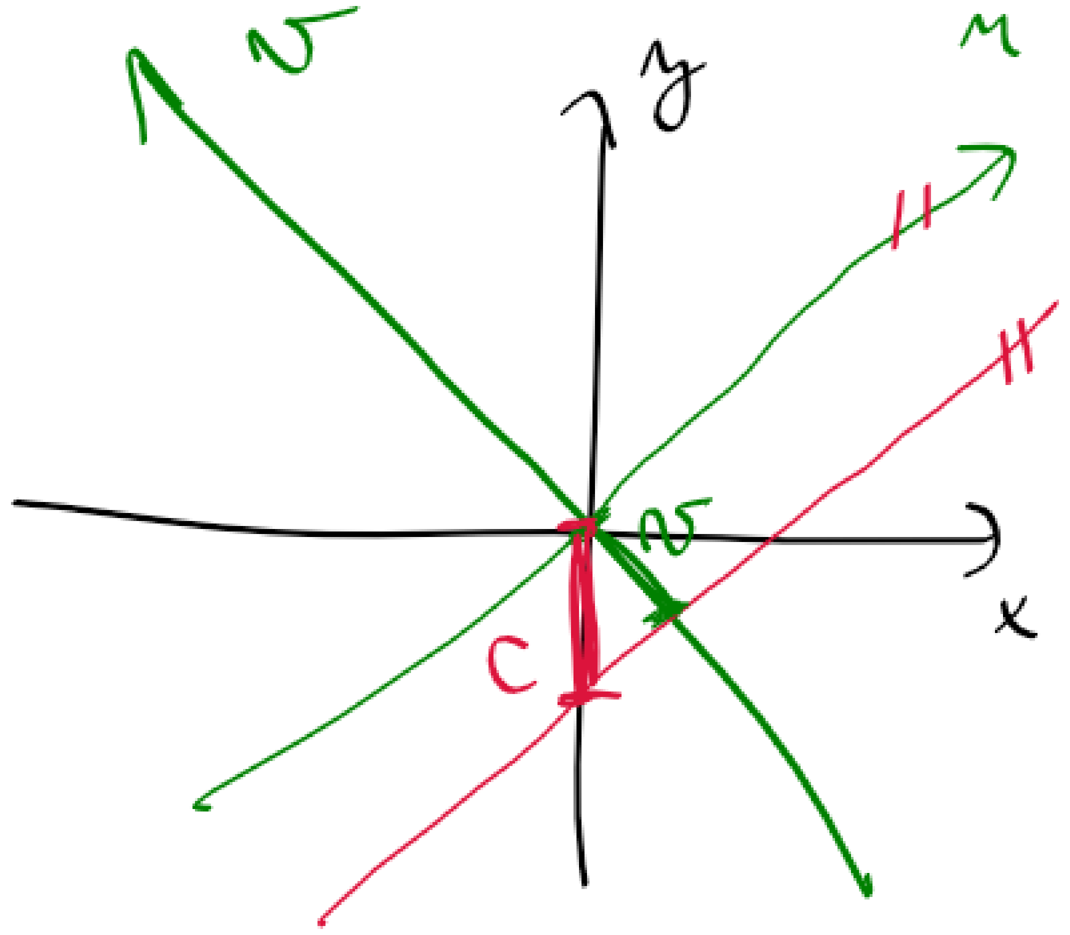
$p: \quad y = c - x$

$$2u^2 = c^2$$
$$u = \frac{c}{\sqrt{2}}$$



$$u = \frac{c}{\sqrt{2}}$$

$$\boxed{u = \frac{y+x}{\sqrt{2}}}$$



$y = x - c$

$$v = \frac{c}{\sqrt{2}} \qquad c = x - y$$

$$\boxed{v = \frac{x-y}{\sqrt{2}}}$$

V RŮZNÝCH SOUŘ.

Mějme $y^2 - x^2 = 1$, tj.

$$u + v = \frac{2x}{\sqrt{2}} \Rightarrow \boxed{x = \frac{u+v}{\sqrt{2}}}$$

$$u - v = \frac{2y}{\sqrt{2}} \Rightarrow \boxed{y = \frac{u-v}{\sqrt{2}}}$$

$$\left(\frac{u-v}{\sqrt{2}}\right)^2 - \left(\frac{u+v}{\sqrt{2}}\right)^2 = 1$$

$$\frac{1}{2}\left(\cancel{u^2} - 2uv + \cancel{v^2} - \cancel{u^2} - 2uv - \cancel{v^2}\right) = 1$$

$$\frac{1}{2}\left(-4uv\right) = 1$$

$$-2uv = 1$$

Tj. $v = \dfrac{1}{-2u}$

Mějme $y = \dfrac{1}{x}$ v novém systému:

Převeďte do souřadnic u, v. Cv.